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Interior solution of a tangentially stressed cylinder in general relativity

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Abstract. A general interior solution of Einstein's field equations for a tangentially stressed cylinder is presented. It is regular everywhere inside a cylinder of radius a and across the surface of the cylinder it is joined smoothly to the exterior Marder solution. The solution is subject to the inequality $C<1$, where C is a parameter nearly equal to twice the mass per unit length of the cylinder. Under certain conditions this solution may be interpreted as the field inside the Raychaudhuri–Som cylindrical cluster of particles.

1. Introduction

In a well known paper Marder (1958) gave the following exterior solution for a cylindrical body:

$$ds^2 = r^{2C}\,dt^2 - r^{2(1-C)}\,d\phi^2 - A^2 r^{2C(C-1)}(dr^2+dz^2), \tag{1}$$

where C is a parameter almost equal to twice the mass per unit length of the body and A is connected not only with C but also, for a cylinder of finite cross section, with the distribution of matter in the cylinder.

In the same paper Marder gave a particular solution for a tangentially stressed cylinder. It is the purpose of this paper to present a general interior solution for a tangentially stressed cylinder. It appears that Marder's solution is a special case of this solution. Our solution is regular everywhere inside a cylinder of radius a and across the surface of the cylinder it is joined smoothly to the exterior Marder solution (1). The tangential stress is positive throughout the body. The solution is, however, subject to the inequality $C<1$. In the particular case when $C<\frac{1}{2}$ it is shown that this solution may be interpreted as the field inside a Raychaudhuri–Som cylindrical cluster (Raychaudhuri and Som 1962).

2. The field equations and their solution

We start with the general static cylindrically symmetric line element:

$$ds^2 = -g_{11}\,dr^2 - g_{22}\,d\phi^2 - g_{33}\,dz^2 + g_{44}\,dt^2 \tag{2}$$

where r and z are the radial and axial coordinates and ϕ is the angular coordinate. The $g_{\mu\nu}$ are, owing to the symmetry of the problem, functions of r only. By a simple transformation of r we can make $g_{11}=g_{33}$. Again, as we are considering only the tangential stress, we shall have $T_1^{\ 1}+T_3^{\ 3}=0$, since both of them are assumed to be absent here. Hence the line element must be of Weyl canonical form (Synge 1966):

$$ds^2=-e^{2\beta-2\alpha}(dr^2+dz^2)-r^2\,e^{-2\alpha}\,d\phi^2+e^{2\alpha}\,dt^2 \tag{3}$$

where α and β are functions of r only.

With the line element (3) the field equations are

$$0=\frac{\beta_1}{r}-\alpha_1^2 \tag{4}$$

$$8\pi p_\phi=e^{2\alpha-2\beta}(\beta_{11}+\alpha_1^2) \tag{5}$$

$$8\pi\rho=e^{2\alpha-2\beta}\left(2\alpha_{11}-\beta_{11}+\frac{2\alpha_1}{r}-\alpha_1^2\right) \tag{6}$$

where the subscript 1 denotes differentiation with respect to r, p_ϕ is the tangential stress and ρ the mass density. A solution of these equations is sought satisfying the following conditions:

(i) $\beta\to 0$ as $r\to 0$ (elementary flatness);

(ii) α, α_1, β and β_1 are continuous across the boundary surface ($r=a$);

(iii) ρ is positive and finite.

Equation (4) shows that if we write

$$\alpha_1=c(r)/r \tag{7}$$

then we must have

$$\beta_1=c^2(r)/r \tag{8}$$

where $c(r)$ is a function of r. Continuity of α_1 and β_1 across the boundary ($r=a$) leads to the interpretation of $c(r)$ as the parameter almost equal to twice the mass per unit length of a cylinder of radius r so that $c(a)=C$.

If we integrate (7) and (8) satisfying the continuity of α and β across the boundary we obtain the following results:

$$\alpha=\int_a^r\frac{c(r)}{r}\,dr+C\ln a \tag{9}$$

$$\beta=\int_a^r\frac{c^2(r)}{r}\,dr+C^2\ln a+\ln A. \tag{10}$$

Thus the line element (3) takes the form

$$ds^2=-A^2a^{2C(C-1)}\left[\exp\left(2\int_a^r\frac{c(r)}{r}(c(r)-1)\,dr\right)\right](dr^2+dz^2)$$
$$-r^2a^{-2C}\left[\exp\left(-2\int_a^r\frac{c(r)}{r}\,dr\right)\right]d\phi^2+a^{2C}\left[\exp\left(2\int_a^r\frac{c(r)}{r}\,dr\right)\right]dt^2. \tag{11}$$

On substituting from (7)–(10) in (5) and (6)

$$8\pi p_\phi = A^{-2}a^{-2C(C-1)}\left[\exp\left(-2\int_a^r \frac{c(r)}{r}(c(r)-1)\,\mathrm{d}r\right)\right]\frac{2cc_1}{r} \tag{12}$$

$$8\pi\rho = A^{-2}a^{-2C(C-1)}\left[\exp\left(-2\int_a^r \frac{c(r)}{r}(c(r)-1)\,\mathrm{d}r\right)\right]\frac{2c_1}{r}(1-c(r)), \tag{13}$$

where $c_1 = \mathrm{d}c(r)/\mathrm{d}r$. Since the metric coefficients (g_{11} etc) as defined in (2) are positive and also since, according to the interpretation of $c(r)$, c_1 is positive, (13) shows that, for ρ to be positive

$$c(r) < 1. \tag{14}$$

At the boundary this leads to the condition

$$C < 1. \tag{15}$$

Further, for ρ to be finite at $r = 0$, we must have $c_1 \propto r$, i.e. $c(r) \propto r^2$ as $r \to 0$.

From the above consideration it is clear from (12) that p_ϕ is always positive and, for ρ to be finite at $r = 0$, tends to zero as $r \to 0$.

It is obvious that the condition of $c(r)$ varying as r^2 as $r \to 0$ makes the metric coefficients regular everywhere.

Now we find M, the mass per unit length of the cylinder. The element of proper cross-sectional area is $r\,\mathrm{e}^{\beta-2\alpha}\,\mathrm{d}r\,\mathrm{d}\phi$ so that the inertial mass contained in unit proper length is

$$M = 2\pi\int_0^a \rho r\,\mathrm{e}^{\beta-2\alpha}\,\mathrm{d}r.$$

On substituting for ρ from (13), this reduces to

$$M = \tfrac{1}{2}\int_0^a \mathrm{e}^{-\beta}c_1(1-c(r))\,\mathrm{d}r = \tfrac{1}{2}C - \tfrac{1}{4}C^2 \tag{16}$$

to zero order in β, irrespective of the distribution of matter in the cylinder.

We now consider a particular solution with $c(r)$ in the simplest form, namely,

$$c(r) = kr^2 \tag{17}$$

where k is an arbitrary constant. Making this substitution in (9) and (10)

$$\alpha = \tfrac{1}{2}k(r^2 - a^2) + C\ln a \tag{18}$$

$$\beta = \tfrac{1}{4}k^2(r^4 - a^4) + C^2\ln a + \ln A. \tag{19}$$

Since we should have $\beta = 0$ at $r = 0$, we have from (19)

$$A = \mathrm{e}^{C^2/4}a^{-C^2}. \tag{20}$$

This solution is the same as the one obtained by Marder for a tangentially stressed cylinder.

3. Field inside a Raychaudhuri–Som cylindrical cluster of particles

Raychaudhuri and Som (1962) discussed the case of a cylindrically symmetric cluster of particles randomly moving in circles perpendicular to the axis of symmetry. We shall now show that under certain conditions the solution obtained in § 2 may be interpreted as the field inside such a cylindrical cluster.

Raychaudhuri and Som took a line element of the form (3) and obtained field equations as follows:

$$0=\frac{\beta_1}{r}-\alpha_1^2 \tag{21}$$

$$8\pi\bar{\rho}\frac{r\alpha_1}{1-2r\alpha_1}=\mathrm{e}^{2\alpha-2\beta}(\beta_{11}+\alpha_1^2) \tag{22}$$

$$8\pi\bar{\rho}\frac{1-r\alpha_1}{1-2r\alpha_1}=\mathrm{e}^{2\alpha-2\beta}\left(2\alpha_{11}-\beta_{11}+\frac{2\alpha_1}{r}-\alpha_1^2\right) \tag{23}$$

where $\bar{\rho}$ is the matter density in the cluster. Now assuming that the interior metric for the cluster is the same as the metric (11) and comparing the field equations (4)–(6) and (21)–(23) we find that the stress–energy tensors in both cases are identical provided

$$\bar{\rho}=\rho\frac{1-2c(r)}{1-c(r)}. \tag{24}$$

It then follows that the solution of § 2 may be interpreted as the field inside that Raychaudhuri–Som cluster whose density is given by (24). One condition has, however, to be satisfied for this identification to be valid. For $\bar{\rho}$ to be positive, we must have, from (24)

$$c(r)<\tfrac{1}{2}. \tag{25}$$

At the boundary this leads to the condition

$$C<\tfrac{1}{2}. \tag{26}$$

Proceeding as before, the mass per unit length of the cylindrical cluster is

$$\bar{M}=2\pi\int_0^a \bar{\rho}r\,\mathrm{e}^{\beta-2\alpha}\,\mathrm{d}r=\tfrac{1}{2}(C-C^2) \tag{27}$$

to zero order in β, irrespective of the distribution of matter in the cluster.

References

Marder L 1958 *Proc. R. Soc.* A **244** 524
Raychaudhuri A K and Som M 1962 *Proc. Camb. Phil. Soc.* **58** 338
Synge J L 1966 *Relativity: The General Theory* (Amsterdam: North Holland)